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Spatially growing wave trails of an inviscid fluid discontinuity

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The eigenvalue equation for three-dimensional waves in parallel and cross flows parallel to a fluid discontinuity has been considered for spatially growing waves. The discontinuity plane (x, y plane) is perpendicular to the gravitational acceleration and consists in general of a jump in speed, in flow direction and in density. With the assumption of waves which are periodic in time and periodic in the y direction, the eigenvalue equation is solved for the complex wavenumber α in the x direction. These waves are used to Fourier synthesize the wave trails generated by a time-periodic disturbance with a Gaussian amplitude distribution $e^{-\lambda y^2}$ along the y axis. Lines of constant phase and lines of constant amplitude within the wave trail have been illustrated for some examples.

1. Introduction

Consider an inviscid and incompressible flow field in Cartesian co-ordinates x, y and z (figure 1). The fluid has a constant density ρ_1 and a constant speed $u_1 > 0$ in the half-space $z \geqslant 0$ and it has a constant density ρ_2 and constant speed $u_2 > 0$ in the half-space $z < 0$. The velocities in the upper and lower half-spaces are perpendicular to the z axis and inclined at angles θ_1 and θ_2 to the x direction. The fluid discontinuity in the x, y plane is perpendicular to the gravitational acceleration $\mathbf{g}$ and this acceleration is antiparallel to the z axis. Such a cross flow is in general unstable against small disturbances. Consider a small displacement $\eta(x = 0) \sim \exp(-\lambda y^2 + i\gamma t)$ of the discontinuity surface in the z direction at the y axis, where $\gamma > 0$ is a given circular frequency, t is time and λ is a positive constant. The above displacement is periodic in time and has a significant amplitude only within a finite range at the y axis; i.e. the displacement is concentrated near the origin of the co-ordinate y. A Gaussian amplitude distribution at the y axis represents an especially simple example of a concentrated disturbance. As a consequence of this excitation at the y axis one intuitively expects the development of a wave pattern which grows within a strip in some direction away from the y axis, where the excitation originates, as indicated in figure 1, and this is confirmed by the results in § 4. The resulting time-periodic and spatially growing wave pattern will be referred to as spatially growing wave trail.

Spatially growing wave trails are interesting since they might be found in any unstable shear flow into which a localized periodic disturbance has been introduced. Local vibrations of the wall below a boundary layer might give rise to similar wave patterns. Wave trails might occur in meteorological configurations.

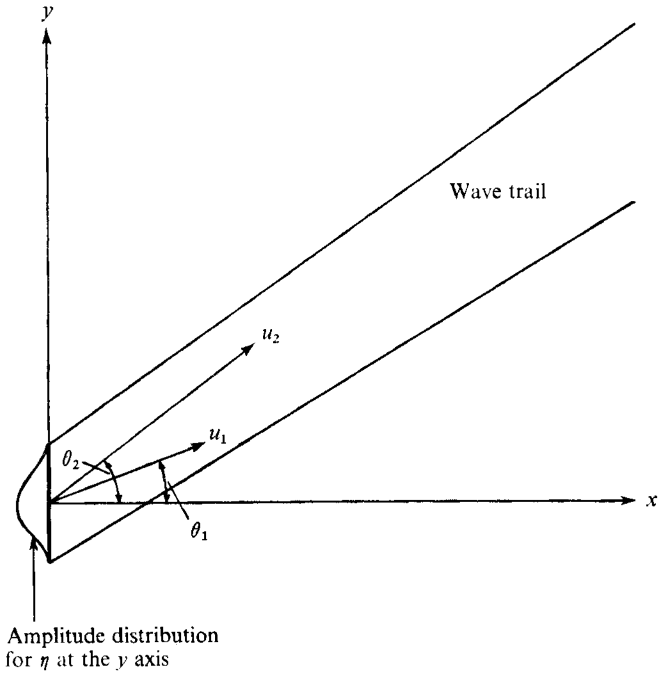

FIGURE 1. Fluid discontinuity and wave trail: top view.

Wavy cloud patterns confined within a trail have been observed occasionally. Shear layers have a finite thickness but the limiting case of an inviscid fluid discontinuity should provide some insight into the development of wave trails.

The stability of an inviscid fluid discontinuity in a parallel flow has been considered by Kelvin (1910) for temporally growing waves. The wave trails considered have to be Fourier synthesized from spatially growing waves which are periodic in time and in the y direction and which show a growth rate in the x direction. Spatially growing waves have been considered only recently by Gaster (1962, 1965), Michalke (1965), Maslowe & Thomson (1971) and Mattingly & Criminale (1971) for various parallel flows. The physical significance of these waves has been demonstrated by Freymuth (1966), Mattingly & Criminale (1972) and Davey & Roshko (1972).

The neutral stability of a fluid discontinuity in a cross flow has recently been considered by Agrawal & Agrawal (1969). A cross flow, however, can be transformed into a parallel flow by means of a Galilean transformation and since a neutral wave remains neutral if seen from a moving reference frame (in which the cross flow appears parallel) physically nothing new is added. On the other hand a spatially growing wave in a cross flow, if seen from the moving reference frame, will appear to grow in space as well as in time and thus cannot be inferred from a spatially growing wave in parallel flow. Hence an interesting aspect is added by a cross flow.

2. The eigenvalue equation

The eigenvalue equation for the stability of an inviscid fluid discontinuity in a parallel flow ($\theta_1 = \theta_2 = 0$) as derived in Lamb's (1932) book can be extended for a cross flow. For a displacement of the form

$$\eta = \exp[i(\alpha x + \beta y + \gamma t)], \tag{1}$$

where α and β are the wavenumbers in x and y directions, the eigenvalue equation for temporally and spatially growing waves reads

$$\alpha^2 + 2\alpha(a_1\beta + a_2\gamma) + b_1\beta^2 + b_2\gamma^2 + 2b_3\beta\gamma + 2c\delta = 0. \tag{2}$$

$a_1 = (\rho_1 u_1^2 \sin\theta_1 \cos\theta_1 + \rho_2 u_2^2 \sin\theta_2 \cos\theta_2)/h$; $a_2 = (\rho_1 u_1 \cos\theta_1 + \rho_2 u_2 \cos\theta_2)/h$; $b_1 = (\rho_1 u_1^2 \sin^2\theta_1 + \rho_2 u_2^2 \sin^2\theta_2)/h$; $b_2 = (\rho_1 + \rho_2)/h$; $b_3 = (\rho_1 u_1 \sin\theta_1 + \rho_2 u_2 \sin\theta_2/h$; $c = (\rho_1 - \rho_2)g/2h$; $h = (\rho_1 u_1^2 \cos^2\theta_1 + \rho_2 u_2^2 \cos^2\theta_2$; and $\delta = (\alpha^2 + \beta^2)^{\frac{1}{2}}$ with $\operatorname{Re}\delta > 0$. If h approaches zero (2) has to be multiplied by h before further evaluation. Solving (2) for α in case of negligible gravity effects and for $h \neq 0$ yields

$$\begin{aligned}\alpha(c=0) = \alpha_0 = -(a_1\beta + a_2\gamma)& \\ \pm i[b_1\beta^2 + b_2\gamma^2 + 2b_3\beta\gamma - (a_1\beta + a_2\gamma)^2]^{\frac{1}{2}}&.\end{aligned} \tag{3}$$

Taking small gravity effects into account yields approximately

$$\alpha \simeq \alpha_0 - c(\alpha_0^2 + \beta^2)^{\frac{1}{2}}/(\alpha_0 + a_1\beta + a_2\gamma). \tag{4}$$

The above waves grow in the propagation direction if the sign in front of the square root in (3) is properly chosen. Equations (3) and (4) are too general for a meaningful discussion. The dependence of α on the three-dimensionality of the waves, the effect of a cross flow and the effect of gravity will be discussed later by means of some examples.

3. Fourier synthesis of wave trails

The wave trail resulting from a time-periodic disturbance

$$\eta(x=0) = \exp(-\lambda y^2 + i\gamma t)$$

with Gaussian amplitude distribution and given frequency γ will be Fourier synthesized from the waves considered in § 2. The representation of $\eta(x=0)$ as a Fourier integral is

$$\eta(x=0) = (4\pi\lambda)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp[-\beta^2/4\lambda + i\beta y]\, e^{i\gamma t}\, d\beta. \tag{5}$$

Consequently one obtains a dependence on x of

$$\eta(x) = (4\pi\lambda)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp[-\beta^2/4\lambda + i\alpha x + i\beta y]\, e^{i\gamma t}\, d\beta. \tag{6}$$

Since the exponent $f(\beta) = -\beta^2/4\lambda + i\alpha x$ is in general a complicated function, an analytic evaluation of the Fourier integral is not possible. For an approximate evaluation $f(\beta)$ will be expanded into a Taylor series around the value $\beta = \beta_0$ at which the real part of $f(\beta)$ has a maximum and only terms up to second order in β will be taken into account. Thus

$$f(\beta) = -\beta^2/4\lambda + i\alpha x \simeq f + Lf'^2 + (\beta - \beta_0 - 2Lf')^2/4L, \tag{7}$$

where $\quad f = f(\beta = \beta_0), \quad f' = \dfrac{\partial f}{\partial \beta}(\beta = \beta_0), \quad L = -\dfrac{1}{2}\left[\dfrac{\partial^2 f}{\partial \beta^2}(\beta = \beta_0)\right]^{-1}.$

Inserting (7) into (6) yields an approximate result for η:

$$\eta = (L/\lambda)^{\frac{1}{2}} \exp\left[i(\beta_0 y + \gamma t) + f - L(y - if')^2\right]. \tag{8}$$

Fourier synthesis of waves has recently been applied by Benjamin (1961), Criminale & Kovasznay (1962) and Gaster & Davey (1968) to a pulsed point disturbance in parallel shear flows.

4. Some examples

Since the wave-trail equation (8) involves many parameters its application and consequences will be discussed by means of specific examples. Other examples can be treated similarly.

4.1. *Parallel flow, gravity neglected* ($\theta_1 = \theta_2 = \theta \neq \frac{1}{2}\pi$, $c = 0$)

It follows from (3) that

$$\alpha = -\beta \tan\theta - \frac{\rho_1 u_1 + \rho_2 u_2}{\rho_1 u_1^2 + \rho_2 u_2^2}\frac{\gamma}{\cos\theta} \pm i(\rho_1\rho_2)^{\frac{1}{2}}\frac{u_1 - u_2}{\rho_1 u_1^2 + \rho_2 u_2^2}\frac{\gamma}{\cos\theta}. \tag{9}$$

Equation (9) shows that the spatial growth rate is independent of β and that velocities enter the growth rate as the velocity difference $u_1 - u_2$ but also appear in the sum of energy densities $\rho_1 u_1^2 + \rho_2 u_2^2$. To calculate the wave trail we need

$$f(\beta) = -\frac{\beta^2}{4\lambda} \mp x(\rho_1\rho_2)^{\frac{1}{2}}\frac{u_1 - u_2}{\rho_1 u_1^2 + \rho_2 u_2^2}\frac{\gamma}{\cos\theta} - ix\left(\beta\tan\theta + \frac{\rho_1 u_1 + \rho_2 u_2}{\rho_1 u_1^2 + \rho_2 u_2^2}\frac{\gamma}{\cos\theta}\right).$$

The real part of $f(\beta)$ has a maximum at $\beta = \beta_0 = 0$. Consequently

$$f = \mp (\rho_1\rho_2)^{\frac{1}{2}}\frac{u_1 - u_2}{\rho_1 u_1^2 + \rho_2 u_2^2}\frac{\gamma x}{\cos\theta} - i\frac{\rho_1 u_1 + \rho_2 u_2}{\rho_1 u_1^2 + \rho_2 u_2^2}\frac{\gamma x}{\cos\theta},$$

$f' = -ix\tan\theta$ and $L = \lambda$. Using (8) the final result is

$$\eta = \exp\left[i\gamma\left(t - \frac{\rho_1 u_1 + \rho_2 u_2}{\rho_1 u_1^2 + \rho_2 u_2^2}\frac{x}{\cos\theta}\right) \mp (\rho_1\rho_2)^{\frac{1}{2}}\frac{u_1 - u_2}{\rho_1 u_1^2 + \rho_2 u_2^2}\frac{\gamma x}{\cos\theta} - \lambda(y - x\tan\theta)^2\right]. \tag{10}$$

To ensure that the wave grows in the propagation direction the positive sign in front of the growth rate applies if $u_1 - u_2 > 0$, otherwise the negative sign applies.

From (10) it follows that the lines of constant phase are parallel to the y axis and they move with constant phase speed in the x direction. Lines of constant amplitude are parabolas with their symmetry axes showing in the flow direction. These results are illustrated in figure 2 for $\rho_1 = \rho_2$, $u_2 = 0$, $\theta = \frac{1}{4}\pi$ and $\gamma/u_1\lambda^{\frac{1}{2}} = 2^{-\frac{1}{2}}$ at a time $t = 0$. In this case

$$\eta(t = 0) = \exp\left[-ix' + x' - (y' - x')^2\right], \tag{11}$$

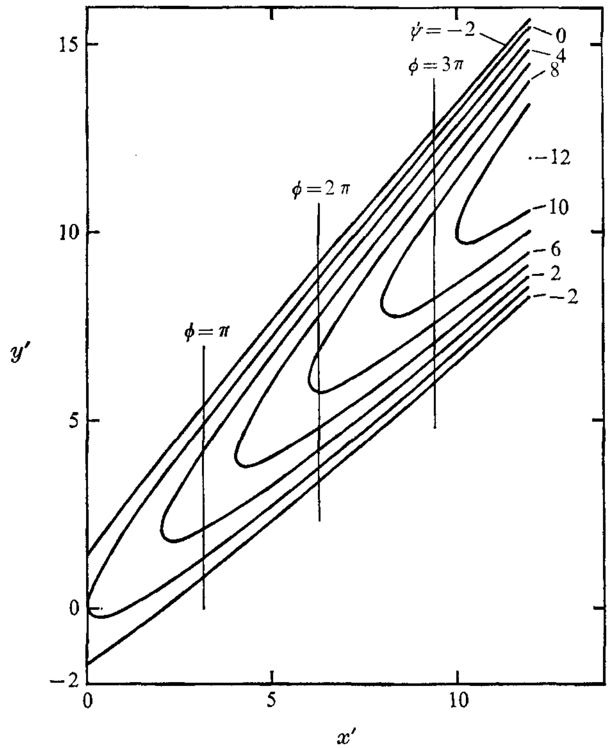

FIGURE 2. Lines of constant phase ϕ and of constant amplitude function ψ of the wave trail, $\rho_1 = \rho_2$, $u_2 = 0$, $\theta = \frac{1}{4}\pi$, $\gamma/u_1\lambda^{\frac{1}{2}} = 2^{-\frac{1}{2}}$ at a time $t = 0$.

where $x' = \lambda^{\frac{1}{2}}x$ and $y' = \lambda^{\frac{1}{2}}y$ are dimensionless co-ordinates. Lines of constant phase $\phi = x'$, and lines of constant amplitude function $\psi = x' - (y - x')^2$ are shown. If amplitudes smaller than e^{-2} of the wave trail are considered as insignificant the wave trail is bounded by a parabola $\psi = -2 = x' - (y' - x')^2$. Hence the wave trail is limited to a strip which spreads parabolically in the flow direction.

4.2. *The effect of a cross flow*

Consider the simple cross flow $u_1 = u_2 = u$, $\rho_1 = \rho_2 = \rho$, $\theta_1 = -\theta_2 = \theta \mp \frac{1}{2}\pi$. It follows from (3) that

$$\alpha = -\gamma/u\cos\theta \pm i\beta\tan\theta. \tag{12}$$

For waves growing in the propagation direction the minus sign in front of $i\beta\tan\theta$ applies if $\beta\sin\theta > 0$, otherwise the positive sign applies. The growth rate is determined by the wavenumber β, i.e. only three-dimensional waves are growing, owing to the discontinuity in the y component of velocity. For two-dimensional waves ($\beta = 0$) the growth rate is zero, owing to the absence of a discontinuity in the x component of velocity.

To calculate the wave trail we need

$$f(\beta) = -\frac{\beta^2}{4\lambda} \mp \beta x\tan\theta - i\frac{\gamma x}{u\cos\theta},$$

$$\beta_0 = \mp 2\lambda x\tan\theta, \quad f = \lambda x^2\tan^2\theta - i\gamma x/u\cos\theta, \quad f' = 0 \quad \text{and} \quad L = \lambda.$$

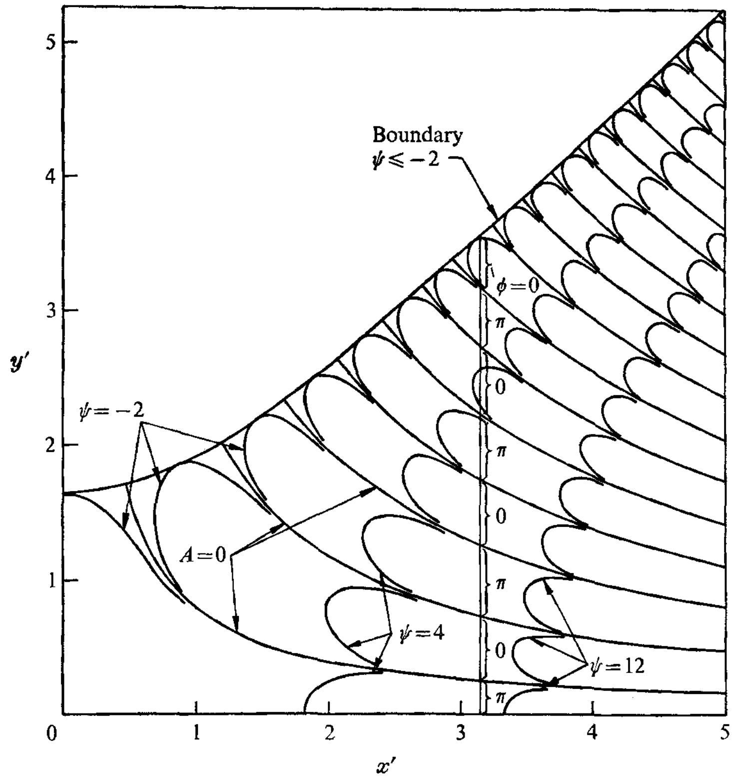

FIGURE 3. Lines of constant phase ϕ and of constant amplitude $A = e^{\psi}$ of the wave trail. $\rho_1 = \rho_2$, $u_1 = u_2 = u$, $\theta_1 = -\theta_2 = \frac{1}{4}\pi$, $\gamma/u\lambda^{\frac{1}{2}} = 2^{-\frac{1}{2}}$ at a time $t = 0$.

We obtained two maxima of equal significance. Calculating η for each maximum from (8) and summing yields

$$\eta = 2\cos(2\lambda xy\tan\theta)\exp i\gamma\left[\left(t-\frac{x}{u\cos\theta}\right)+\lambda\tan^2\theta x^2-\lambda y^2\right]. \tag{13}$$

Summing the results for both maxima gives a reasonable approximation only if the maxima are sufficiently separated from eac hother, i.e. for large x. An analytic representation for the region near the origin of the wave trail is not possible.

Equation (13) represents a wave trail with lines of constant phase parallel to the y axis which move with a phase speed $u\cos\theta$ in the x direction. Lines of constant amplitude are transcendental functions. These results are illustrated in figure 3 for $\theta = \frac{1}{4}\pi$ and $\gamma/u\lambda^{\frac{1}{2}} = 2^{-\frac{1}{2}}$ at a time $t = 0$. In this case

$$\eta(t=0) = 2\cos 2x'y'\exp(-ix'+x'^2-y'^2). \tag{14}$$

Since the y' axis is a symmetry axis, representation will be restricted to one quadrant. If amplitudes smaller than e^{-2} are considered as insignificant the wave trail is bounded by a hyperbola $y'^2-x'^2 = 2+\ln 2$, i.e. for large x the wave trail is concentrated between straight lines $x' = \pm y'$ parallel to the two cross-flow directions. The lines of constant amplitude $A = e^{\psi} = 2\cos 2x'y'\exp(x'^2-y'^2)$ represent an elaborate interference pattern.

4.3. *The effect of gravity*

To demonstrate the effect of gravity, consider the simple case where $\theta_1 = \theta_2 = \theta$, $u_2 = 0$ and $c < 0$. It follows from (4) that

$$\alpha \simeq -\gamma/u_1 + c(1-\beta^2 u_1^2/4\gamma^2) - i[\gamma/u_1 + c(1+\beta^2 u_1^2/4\gamma^2)]. \qquad (15)$$

Equation (15) shows that when the lighter fluid is on top of the heavier one ($c < 0$) the spatial amplification rate is decreased by gravity. Three-dimensionality ($\beta \neq 0$) decreases the amplification rate further.

To calculate the wave trail we need

$$f(\beta) = -\beta^2/4\lambda + [\gamma/u_1 + c(1+\beta^2 u_1^2/4\gamma^2)]x + i[-\gamma/u_1 + c(1-\beta^2 u_1^2/4\gamma^2)]x,$$
$$\beta_0 = 0, \quad f = (1+cu_1/\gamma)\gamma x/u_1 + i(-1+cu_1/\gamma)\gamma x/u_1, \quad f' = 0,$$
$$L = \lambda \frac{1 - cu_1^2\lambda x/\gamma^2 - icu_1^2\lambda x/\gamma^2}{1 - 2cu_1^2\lambda x/\gamma^2 + 2(cu_1^2\lambda x/\gamma^2)^2}.$$

To limit the discussion we consider only the asymptotic behaviour of the wave-trail amplitude A for large values of x, for which we obtain

$$A \simeq [\gamma/u_1(2^{\frac{1}{2}}\lambda cx)^{\frac{1}{2}}] \exp(1+cu_1/\gamma)\gamma x/u_1 + \gamma^2 y^2/2cu_1^2 x. \qquad (16)$$

Lines of constant amplitude are nearly hyperbolas; i.e., owing to gravity the wave trail spreads faster to the sides than without gravity (parabolic spreading).

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